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STEPHEN J. MADDEN, JR.

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ABSTRACT

The earth's atmosphere is considered as made up of oblate spheroidal layers of variable density lying over an oblate spheroidal Earth. The gravitational attraction of the atmosphere at exterior points is computed and its contribution to the usual spherical harmonic gravitational expansion is assessed. The potential is also found for points at the bottom of the model atmosphere. This latter result is of interest for determination of the potential at the surface of the geoid. The atmospheric correction to the geoid determined from satellite coefficients is given.

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I. Introduction

The gravitational contribution for the earth's atmosphere is becoming of more interest as the precision of gravitational determinations increases and as it becomes important to relate satellite altimetry measurements to the gravitational field of earth. Accordingly, it is useful to have estimates of the importance of its effect. The present report is concerned with a simplified treatment which is analytically tractable and which, at the same time, preserves the essential features of the actual situation.

Since there is a flattening effect due to the ellipticity of the earth, the atmosphere is considered to be made up of oblate spheroidal layers of constant density. The layers are confocal with the reference ellipsoid defined by the adopted reference constants of the 1968 I.A.U. The lower limit of the atmosphere is assumed to be the reference ellipsoid.

The general procedure used is to find the contribution, to the external and internal potentials, of a thin spheroidal shell of constant density. This shell is bounded by confocal ellipsoids. The external ellipsoid has semi-major axis a' , semi-minor axis c' and eccentricity e' ; the corresponding quantities for the internal ellipsoid are a , c , and e . Since the ellipsoids are confocal,

$$a'e' = ae = \sqrt{a^2 - c^2} = \sqrt{a'^2 - c'^2} \triangleq E \quad (1)$$

Once the potential of the shell is found, a limiting argument is used

to find the contribution of a shell of finite thickness and variable density.

2. The External Potential

The solution to the external problem follows easily from a result in (Macmillan, 1958). If the field point is located by spherical coordinates r and θ outside of a solid ellipsoid of density ρ , then the potential is

$$V = 3MG \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} \frac{E^{2n}}{r^{2n+1}} P_{2n}(\cos \theta), \quad (2)$$

where M is the mass of the spheroid, G is the gravitational constant, r is the distance from the center, and θ is the colatitude. If the potential due to a spheroid of density ρ and semi-major axis a' is computed using (2), and if the potential due to a spheroid of semi-major axis a and density $-\rho$ is computed, then the potential caused by the spheroidal shell contained between the two bounding surfaces is

$$\Delta V = 3G \Delta M \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} \frac{E^{2n}}{r^{2n+1}} P_{2n}(\cos \theta).$$

Here ΔM is the mass of the shell, and $r > a$. The total potential contribution of a sequence of very thin layers is, using a limiting argument,

$$V_{ATM} = 3GM_A \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} \frac{E^{2n}}{r^{2n+1}} P_{2n}(\cos \theta). \quad (3)$$

The external potential of the atmosphere depends only on the total mass and the linear eccentricity, E . This result is in agreement with the more general MacLaurin's theorem (MacMillan, 1958). In this expression,

the quantity E is the same as that for the reference ellipsoid. It can be seen, from a comparison test using $|P_{2n}(\cos \theta)| < 1$, that this series converges at all points on the ellipsoid.

3. The Internal Potential

The derivation of the internal potential is more complicated. Since there is an axial symmetry in the problem we need only find the potential on the axis of symmetry, the z -axis. (MacMillan, 1958, pg. 360). The first step is to find an expression analogous to (2) where the field point lies inside a homogeneous ellipsoid of density ρ . This expression follows as a special case of an expression in (MacMillan, 1958), for this potential,

$$V = \pi \rho G a^2 c \int_0^\infty \left(1 - \frac{z^2}{c^2+s}\right) \frac{ds}{\sqrt{(a^2+s)^2(c^2+s)}}.$$

The integral here can be evaluated directly in terms of elementary functions (Pierce and Foster, 1961), or found as a limiting case of the triaxial potential (Byrd and Friedman, 1971). It is

$$V = \frac{2\pi \rho G a^2 c}{E} \left\{ \left(1 + \frac{z^2}{E^2}\right) \sin^{-1} e - \frac{z^2}{cE} \right\} \quad (4)$$

Here z lies within the ellipsoid.

If we consider the contribution to the potential of a thin shell between ellipsoids with $a' = a + \Delta$ and a , where $\Delta/a \ll 1$, then if z lies inside both shells,

$$\begin{aligned} \Delta V_{\text{shell}} &= \frac{2\pi G \rho a'^2 c'}{E} \left\{ \left(1 + \frac{z^2}{E^2}\right) \sin^{-1} e' - \frac{z^2}{c'E} \right\} \\ &\quad - \frac{2\pi G \rho a^2 c}{E} \left\{ \left(1 + \frac{z^2}{E^2}\right) \sin^{-1} e - \frac{z^2}{cE} \right\} \end{aligned}$$

or

$$\Delta V_{\text{shell}} = \frac{2\pi\rho G}{E} \left\{ \left(1 + \frac{z^2}{E^2} \right) [a'^2 c' \sin^{-1} e' - a^2 c \sin^{-1} e] - \frac{z^2}{E} (a'^2 - a^2) \right\}.$$

If we look at the case where $\Delta/a \ll 1$, then

$$a'^2 - a^2 \approx 2a\Delta.$$

Similarly if we define

$$f(a) = a^2 c \sin^{-1} e,$$

with

$$c = \sqrt{a^2 - E^2}, \quad e = \frac{E}{a},$$

then

$$f(a') \approx f(a) + f'(a)\Delta.$$

If we carry out the differentiation,

$$a'^2 c' \sin^{-1} e' - a^2 c \sin^{-1} e \approx (2ac \sin^{-1} e + \frac{a^3}{c} \sin^{-1} e - Ea)\Delta.$$

Thus, on the z-axis,

$$\Delta V_{\text{shell}} = \frac{2\pi\rho G}{E} \left\{ \left(1 + \frac{z^2}{E^2} \right) [2ac \sin^{-1} e + \frac{a^3}{c} \sin^{-1} e - Ea]\Delta - \frac{2z^2}{E} a\Delta \right\},$$

or

$$\Delta V_{\text{shell}} = 2\pi G \rho \Delta \left\{ \frac{M}{E} + z^2 \frac{M}{E^3} - \frac{2z^2 a}{E^2} \right\} \quad (5)$$

with

$$M = 2ac \sin^{-1} e \frac{a^3}{c} \sin^{-1} e - Ea.$$

The total contribution of many thin shells to the potential is more cumbersome than in the exterior case since the density variation must be explicitly considered. Let us rewrite (5) as

$$\Delta V_{\text{shell}} = 2\pi G Q(a) \rho(a) da, \quad (6)$$

with

$$da = \Delta$$

and

$$Q(a) = \frac{M}{E} + z^2 \frac{M}{E^3} - \frac{2z^2 a}{E^2}.$$

If we sum the total contributions of many shells, using (6),

$$V_{\text{ATM}} = 2\pi G \int_a^\infty Q(\xi) \rho(\xi) d\xi$$

where a now corresponds to the reference ellipsoid. For present purposes the density will be assumed to be an exponential function of the quantity ξ ,

$$\rho(\xi) = \rho_0 e^{-\frac{\xi-a}{H}}, \quad 5.$$

with H a scale height characteristic of the lower atmosphere where the bulk of the atmospheric mass is located. A typical value for H is 8Km. With this assumption,

$$V_{ATM} \approx 2\pi G\rho_0 \int_a^{\infty} e^{-\frac{\xi-a}{H}} Q(\xi) d\xi.$$

In the evaluation of this integral we can take advantage of the relative sizes of a and H ; the quantity $Q(\xi)$ varies slowly during a scale height variation in ξ and, as can be shown,

$$V_{ATM} \approx 2\pi G\rho_0 Q(a) \int_a^{\infty} e^{-\frac{\xi-a}{H}} d\xi,$$

or

$$V_{ATM} = 2\pi G\rho_0 H Q(a),$$

where terms on the order of H/a have been neglected with respect to one. If we return to the notation of (5),

$$V_{ATM} = 2\pi\rho_0 GH \left(\frac{M}{E} + z^2 \frac{M}{E^3} - \frac{2z^2 a}{E^2} \right). \quad (7)$$

This expression can be evaluated explicitly if the reference values of the constants are used. However, since we are primarily interested in representative values, we can utilize the fact that the eccentricity of the reference ellipsoid is small to find that

$$M = 2a^2 e + \frac{4}{15} a^2 e^5 + O(e^7).$$

If this is used in (7),

$$V_{ATM} = 2\pi\rho_0GH \left[2a + \frac{4}{15} z^2 \frac{e^2}{a} \right].$$

This expression yields the potential anywhere inside the atmospheric shell simply by replacing z^2 with $z^2 - (x^2 + y^2)/2$, and finally

$$V_{ATM} = 2\pi\rho_0GH \left[2a + \frac{4}{15} \frac{e^2}{a} \left(z^2 - \frac{x^2 + y^2}{2} \right) \right]. \quad (8)$$

This equation, plus that of (3) yields the desired potential expressions.

4. Examination of the Solutions

The most natural questions concerning the external gravitational potential of the atmosphere are those involving its contribution to the already measured coefficients of the total external potential of the earth and atmosphere. The models of the atmosphere used here possess symmetry with respect to the rotation axis of the earth and hence can contribute only zonal terms. We thus restrict consideration to these terms. The standard zonal expression is given as

$$V_T = \frac{GM_T}{r} \left\{ 1 - \sum_{n=2}^{\infty} \left(\frac{a}{r} \right)^n J_n P_n(\cos \theta) \right\},$$

or

$$V_T = \frac{GM_T}{r} \left\{ 1 - \left(\frac{a}{r} \right)^2 J_2 P_2(\cos \theta) - \left(\frac{a}{r} \right)^3 J_3 P_3(\cos \theta) - \left(\frac{a}{r} \right)^4 J_4 P_4(\cos \theta) + \dots \right\}$$

with the subscript T referring to the contribution of both the solid earth and atmosphere. If we write the corresponding external contribution to the atmosphere, using (3),

$$V_A = \frac{GM_T}{r} \left\{ \frac{M_A}{M_T} - \frac{e^2}{5} \frac{M_A}{M_T} \left(\frac{a}{r} \right)^2 P_2(\cos \theta) + \frac{3e^4}{35} \frac{M_A}{M_T} \left(\frac{a}{r} \right)^4 P_4(\cos \theta) \dots \right\}$$

With this form we can compare the magnitudes of J_2 and J_4 and the atmospheric terms. If we use the subscript A to denote quantities pertaining to the atmosphere alone, then

$$J_{2A} = - \frac{e^2}{5} \frac{M_A}{M_T}$$

and

$$J_{4A} = \frac{3e^4}{35} \frac{M_A}{M_T}$$

The constants appearing here are the eccentricity, available from the I.A.U. flattening, $f = 1/298.25$, and from (Verniani, 1966) are

$$e^2 = .0066945,$$

$$M_A = 5.136 \times 10^{21} \text{ gm}$$

$$M_T = 5.976 \times 10^{27} \text{ gm}$$

With these values we find

$$J_{2A} = -1.151 \times 10^{-9}$$

$$J_{4A} = 3.302 \times 10^{-11}$$

This value for J_{2A} compares favorably with the value -1.9×10^{-9} arrived at by Lundquist (private communication) who assumed all of the atmospheric mass was concentrated in a shell of the dimensions of the reference ellipsoid. These numbers correspond to the total values

$$J_2 = 1082.7 \times 10^{-6}$$

$$J_4 = -1.649 \times 10^{-6}$$

The atmospheric values enter in the seventh place in J_2 and in the sixth place in J_4 .

It is probable that the actual, non-idealized, atmosphere actually has a larger influence than those shown here. The atmosphere is not of constant density on spheroidal shells, and an examination of data shows that the earth probably has a somewhat larger value of J_{2A} than is shown here.

The case of the interior potential is somewhat different, here there is no convenient series for comparison. For this reason we compute, as a measure of importance, the maximum interior attraction of the atmosphere at a point on the surface of the reference ellipsoid. From equation (8) it follows that, at any point inside the atmospheric shell, the force is

$$\underline{F}_{ATM} = \nabla V_{ATM} = 2\pi\rho_0 GH \left(\frac{4}{15} \frac{e^2}{a} \right) \begin{pmatrix} -x \\ -y \\ 2z \end{pmatrix},$$

or

$$|\underline{F}_{ATM}|^2 = \left(\frac{8}{15} \pi\rho_0 GH \frac{e^2}{a} \right)^2 (x^2 + y^2 + 4z^2).$$

Now on the reference ellipsoid

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1,$$

so that on this surface

$$|F_{\text{ATM}}|^2 = \left(\frac{8}{15} \pi \rho_0 G H \frac{e^2}{a} \right)^2 \left\{ a^2 + \left(4 - \frac{a^2}{c^2} \right) z^2 \right\}.$$

This last quantity varies as a function of z^2 which has the maximum value $z^2 = c^2$. Thus

$$|F_{\text{ATM}}|_{\text{max}} = \frac{16}{15} \pi \rho_0 G H e^2 \frac{c}{a}.$$

If we use the numerical values

$$\rho_0 = 1.23 \times 10^{-3} \text{ gm/cm}^3$$

$$G = 6.67 \times 10^{-8} \text{ cm}^3/\text{gm sec}^2$$

$$H = 8 \times 10^5 \text{ cm}$$

$$e^2 = .0066945$$

$$\frac{c}{a} = 1 - f = 1 - \frac{1}{298.25},$$

then

$$|F_{\text{ATM}}|_{\text{max}} = 1.5 \times 10^{-6} \text{ cm/sec}^2 = 1.5 \times 10^{-3} \text{ mgal.}$$

This result is extremely small. It corresponds to a negligible displacement of the geoid.

To assess the displacement of the geoid caused by the atmosphere, we use the equation which states that it is an equipotential surface of gravity. Let V be the gravity potential not including the atmosphere and $\epsilon \delta V$ be the internal atmospheric potential, then the geoid is defined by

$$V(r(\epsilon), \theta, \phi) + \epsilon \delta V(r(\epsilon), \theta, \phi) = \text{const.} \quad (9)$$

Here ϵ is the ratio of the mass of the atmosphere to the mass of the earth plus the mass of the atmosphere, a small quantity. If ϵ is set to zero we find the usual radius. Assume a Taylor expansion for $r(\epsilon)$, and

$$r(\epsilon) = r(0) + \left(\frac{dr}{d\epsilon} \right)_{\epsilon=0} \epsilon + \dots$$

To first order, the correction to the radius is given by

$$\delta r = r(\epsilon) - r(0) = - \frac{\epsilon \delta V}{g},$$

a result which follows from differentiation of (9).

If we use

$$M_A = 4\pi \rho_0 a^2 H,$$

then (8) becomes

$$V_{\text{ATM}} = \epsilon \delta V = \frac{GM_T}{r} \left(\frac{M_A}{M_T} \right) \left\{ \frac{r}{a} + \frac{2}{15} e^2 \left(\frac{r}{a} \right)^3 P_2(\cos \theta) \right\}$$

and hence

$$\delta r = \frac{-\epsilon \delta V}{g} = \frac{M_A}{M_T} r \left\{ \frac{r}{a} + \frac{2}{15} e^2 \left(\frac{r}{a} \right)^3 P_2(\cos \theta) \right\}$$

To find a representative value, we set

$$r = a, \quad P_2(\cos \theta) = 1, \quad \text{and}$$

$$\delta r = \frac{M_A}{M_T} a \left\{ 1 + \frac{2}{15} e^2 \right\}.$$

The first term corresponds to a simple correction of the mass figure when we are operating inside the atmosphere, the second is a smaller latitude dependent term. To estimate their magnitude we use the

values on the previous pages to find

$$\delta r_1 = \frac{M_A}{M_T} a = 5.5 \text{ meters}$$

and

$$\delta r_2 = \frac{M_A}{M_T} a \left(\frac{2}{15} e^2 \right) \approx .5 \text{ cm.}$$

For present purposes, it is apparent that a simple correction of the mass will suffice. The remaining latitude dependent term is inconsequential with radar altimeter accuracies at their present or anticipated levels.

It is possible, of course, that the term δr_2 above is smaller than the true atmospheric contribution would be if latitudinal density variations were taken into account. However, if the result is small by a factor of ten, this term will not need to be included in analyses until radar altimeter accuracy figures improve by a factor of one hundred.

REFERENCES

1. Byrd, P.F., and Friedman, M.D., Handbook of Elliptic Integrals for Engineers and Scientists, 2nd. ed., Springer-Verlag, New York, 1971.
2. MacMillan, W.D., The Theory of the Potential, Dover, New York, 1958.
3. Heiskanen, W.A., and Moritz, H., Physical Geodesy, Freeman, San Francisco, 1967.
4. Hotine, M., Mathematical Geodesy, ESSA Monograph No. 2, U.S. Department of Commerce, Washington, D.C., 1969.
5. Pierce, B.O., and Foster, R.M., A Short Table of Integrals, Ginn and Co., Boston, 1961.
6. Verniani, F., "The Total Mass of the Earth's Atmosphere," Journal of Geophysical Research, V. 71, No. 2, January 15, 1966.